

A Lagrangian kinetic model for collisionless magnetic reconnection

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Abstract

A new fully kinetic system is proposed for modeling collisionless magnetic reconnection. The formulation relies on fundamental principles in Lagrangian dynamics, in which the electron inertia is neglected in the expression of the Lagrangian, rather than enforcing a zero electron mass in the equations of motion. This is done upon splitting the electron velocity into its mean and fluctuating parts, so that the latter naturally produce the corresponding pressure tensor. If the electron heat flux is neglected, then the strong electron magnetization limit yields a hybrid model for kinetic ions and fluid electrons, in which the electron pressure tensor is frozen into the electron mean velocity.

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1 Introduction

A well known necessary condition for magnetic reconnection in plasmas is that the magnetic field lines are not frozen into the fluid flow. For example, while ideal MHD (with frozen-in magnetic field) fails to reproduce reconnection, a great advance is provided by resistive MHD, in which the frozen-in condition is broken by a finite resistivity. However, although reliable results are obtained by resistive MHD in several conditions, modeling collisionless reconnection requires extra effort, due to the special nature of the kinetic features underlying this phenomenon. Kinetic theory features first make their appearance in the dynamics of electrons, whose pressure tensor is often assumed to dominate over the inertia of their mean flow. In addition, the high energy levels of the ions lead to the necessity of a kinetic treatment also for these particles.

1.1 The kinetic model

In the light of the above arguments, fully kinetic simulations become necessary in modelling collisionless reconnection and they are based on a set of three equations, which may be conveniently written in terms of the ion distribution on phase space $f_i(\mathbf{x}, \mathbf{v})$, the *relative* electron distribution $\tilde{f}_e(\mathbf{x}, \mathbf{c})$ (where $\mathbf{c} = \mathbf{v} - \mathbf{V}_e$ and \mathbf{V}_e is the electron mean velocity), and the magnetic induction field \mathbf{B} . This paper proposes the following model for inertialess electrons:

$$\frac{\partial \tilde{f}_e}{\partial t} + (\mathbf{c} + \mathbf{V}_e) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{x}} + \left[\mathbf{c} \times \left(\frac{q_e}{m_e} \mathbf{B} - \nabla \times \mathbf{V}_e \right) - \mathbf{c} \cdot \nabla \mathbf{V}_e + \frac{1}{m_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{c}} = 0 \quad (1)$$

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left[(\mathbf{v} - \mathbf{V}_e) \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V}_e \times \mathbf{B} - \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right) \quad (3)$$

where

$$n_e = -\frac{q_i}{q_e} \int f_i d^3 \mathbf{v}, \quad q_e n_e \mathbf{V}_e = \nabla \times \mathbf{B} - q_i \int \mathbf{v} f_i d^3 \mathbf{v}, \quad \tilde{\mathbb{P}}_e = m_e \int \mathbf{c} \mathbf{c} \tilde{f} d^3 \mathbf{c} \quad (4)$$

so that $n_e = \int \tilde{f}_e d^3 \mathbf{c}$ is the electron density and $\tilde{\mathbb{P}}_e$ is the electron pressure tensor. Here, the constants q_e and q_i denote the electron and ion charge, respectively, and m_e and m_i denote their corresponding masses. While the above dynamics of the magnetic field (3) and the ion probability density (2) is commonly found in the literature and it is easily derived by neglecting inertial terms in the equation for the electron mean velocity \mathbf{V}_e , the kinetic equation (1) for the relative electron density exhibits the new Coriolis-type force $\mathbf{c} \times \boldsymbol{\omega}_e$, where $\boldsymbol{\omega}_e = \nabla \times \mathbf{V}_e$ is the mean electron vorticity. This type of non-inertial forcing appears typically in moving frames (recall that \tilde{f}_e is the phase-space density in the frame moving with \mathbf{V}_e), as shown in [17] for the case of general magnetized plasmas. However, this term is lost in common models, precisely in the step when one neglects electron inertia, after splitting electron mean and fluctuation velocities. This means that the asymptotic process leading to Hall MHD does not retain important features, such as non-inertial forces, when one aims to account for the fluctuation kinetics. Retaining these important features requires a different method. In particular, equations (1)–(4) are derived in this paper by applying Lagrangian methods: the same methods that yielded MHD in [7] and to Hall MHD in [10].

1.2 Electron pressure tensor dynamics

In a series of papers (see e.g. [19, 22, 13, 14, 21] and subsequent papers by the same authors), considerable effort has been dedicated to the question of whether a moment truncation is possible, that would give a satisfactory description of electron kinetics. Such a moment truncation must account for the pressure tensor dynamics, so that the simplest possible truncation would rely on the assumption

of a negligible electron heat flux. Under this hypothesis, equation (1) eventually yields

$$\frac{\partial \tilde{\mathbb{P}}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) \tilde{\mathbb{P}}_e + (\nabla \cdot \mathbf{V}_e) \tilde{\mathbb{P}}_e + \frac{q_e}{m_e} (\mathbf{B} \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \mathbf{B}) + \tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e + (\tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e)^T - \underbrace{(\boldsymbol{\omega}_e \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \boldsymbol{\omega}_e)}_{\text{Coriolis force terms}} = 0. \quad (5)$$

Here, the new Coriolis term persists as it gives non-zero contribution in the pressure tensor dynamics. More importantly, one observes that in the strong electron magnetization limit $\mathbf{B} \times \tilde{\mathbb{P}} - \tilde{\mathbb{P}} \times \mathbf{B} \simeq 0$, the electron pressure tensor density is *frozen* into the electron mean velocity, that is

$$\frac{d}{dt} \left(\tilde{\mathbb{P}}_{jk}(\mathbf{q}_t, t) dq_t^j dq_t^k d^3 \mathbf{q}_t \right) = 0 \quad \text{along} \quad \frac{d\mathbf{q}_t}{dt} = \mathbf{V}_e(\mathbf{q}_t, t).$$

This means that the electron velocity correlations are constant along the electron mean flow. Although, this is a nice picture, we should not forget that it assumes a negligible electron heat flux. The question whether this assumption is justifiable is still open, with results that basically depend on the particular situation that is being considered each time [14].

1.3 Main content of the paper

1. Section 2 formulates the model by applying the hypothesis of negligible inertia for the electron mean flow. Instead of inserting this hypothesis in the equations of motion, this is done in the variational framework by using standard techniques in the theory of continuum systems with Lagrangian labels [7].
2. Section 3 analyzes the consequences of the model formulated in Section 2 in terms of the electron pressure tensor. It is shown how the strong electron magnetization limit yields the frozen-in condition for the electron pressure tensor, as long as the heat flux contribution may be neglected. Also, it is emphasized how this new frozen-in law for the electron pressure can be used to construct new hybrid models, which discard the information about higher order moments of the electron distribution.
3. Section 4 presents the stationary equations in various cases and shows how the present model recovers the Harris current sheet solutions of the Maxwell-Vlasov system. In the same Section, a conserved total energy is presented explicitly, along with two families of constants of motion that are inherited from the Maxwell-Vlasov system.

2 Formulation of the model

The Lagrangian approach to continuum systems has a long standing tradition, whose most famous result is probably Arnold's formulation of Euler's fluid equation [1] in terms of geometry and symmetry. Later, this approach was extended to various compressible fluid models in different contexts [7]. In plasma kinetic theory, the Lagrangian approach appeared in the well known work by Low [12], which then was considered by Dewar in [4]. This approach achieved its most prominent result in Littlejohn's formulation of guiding center motion [11]. Later, this approach was pursued by many others [20, 2, 3]. When the variational approach arises from an action principle that is written in terms of purely Lagrangian labels, then this approach is often known as Euler-Poincaré variational method [7]. For example, the variational formulations of ideal MHD [7] and Hall MHD [10] are exactly of this type. The present approach is mainly inspired by the results in [3, 10].

2.1 The action principle

In order to derive an appropriate action principle for modeling collisionless reconnection, let us write the Eulerian action functional as

$$\begin{aligned} \delta \int_{t_1}^{t_2} & \left(\frac{m_i}{2} \int f_i |\mathbf{u}_i|^2 d^3\mathbf{x} d^3\mathbf{v} + q_i \int f_i \mathbf{u}_i \cdot \mathbf{A} d^3\mathbf{v} d^3\mathbf{x} - q_i \int \varphi f_i d^3\mathbf{v} d^3\mathbf{x} \right. \\ & + \frac{m_e}{2} \int \tilde{f}_e |\tilde{\mathbf{u}}_e + \epsilon \mathbf{V}_e|^2 d^3\mathbf{x} d^3\mathbf{c} + q_e \int \tilde{f}_e (\tilde{\mathbf{u}}_e + \mathbf{V}_e) \cdot \mathbf{A} d^3\mathbf{c} d^3\mathbf{x} - q_e \int \varphi \tilde{f}_e d^3\mathbf{c} d^3\mathbf{x} \\ & \left. - \frac{m_i}{2} \int f_i |\mathbf{u}_i - \mathbf{v}|^2 d^3\mathbf{x} d^3\mathbf{v} - \frac{m_e}{2} \int \tilde{f}_e |\tilde{\mathbf{u}}_e - \mathbf{c}|^2 d^3\mathbf{x} d^3\mathbf{c} - \frac{1}{2} \int |\nabla \times \mathbf{A}|^2 d^3\mathbf{x} \right) dt = 0 \quad (6) \end{aligned}$$

where the notation is as in equations (1)–(4). In addition, (φ, \mathbf{A}) denotes the electromagnetic potentials, $\epsilon \in [0, 1]$ is a convenient parameter and the quantities

$$\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}, \mathbf{v}, t), \quad \tilde{\mathbf{u}}_e = \tilde{\mathbf{u}}_e(\mathbf{x}, \mathbf{c}, t)$$

denote the position components of the Eulerian phase-space velocity for the two species, so that $\dot{\mathbf{x}}_i = \mathbf{u}_i(\mathbf{x}, \mathbf{v}, t)$ for the ions while $\dot{\mathbf{x}}_e = \tilde{\mathbf{u}}_e(\mathbf{x}, \mathbf{c}, t) + \mathbf{V}_e(\mathbf{x}, t)$ for the electrons. In more generality, one defines the six-dimensional phase-space velocities \mathbf{X}_i and $\tilde{\mathbf{X}}_e$ so that, by a slight abuse of notation one may write

$$(\dot{\mathbf{x}}, \dot{\mathbf{v}})_i = (\mathbf{u}_i, \mathbf{a}_i) =: \mathbf{X}_i$$

for the ions and

$$(\dot{\mathbf{x}}, \dot{\mathbf{c}})_e = (\tilde{\mathbf{u}}_e + \mathbf{V}_e, \tilde{\mathbf{a}}_e), \quad \text{with} \quad \tilde{\mathbf{X}}_e := (\tilde{\mathbf{u}}_e, \tilde{\mathbf{a}}_e),$$

for the electrons. Here, the accelerations \mathbf{a}_i and $\tilde{\mathbf{a}}_e$ do not appear in the action principle (6), since the total energy cannot depend on the particle accelerations. Along the lines of [3], the terms in $|\mathbf{u}_i - \mathbf{v}|^2$ and $|\tilde{\mathbf{u}}_e - \mathbf{c}|^2$ in (6) are used to constrain the velocities \mathbf{u}_i and $\tilde{\mathbf{u}}_e$ to the corresponding Eulerian coordinate so that

$$\mathbf{u}_i(\mathbf{x}, \mathbf{v}, t) = \mathbf{v}, \quad \tilde{\mathbf{u}}_e(\mathbf{x}, \mathbf{c}, t) = \mathbf{c} \quad (7)$$

Notice that expanding all terms containing \mathbf{u}_i and $\tilde{\mathbf{u}}_e$ in (6) yields Littlejohn's phase-space Lagrangian [11] for ions and electrons. This occurs because of the minus signs carried by the terms in $|\mathbf{u}_i - \mathbf{v}|^2$ and $|\tilde{\mathbf{u}}_e - \mathbf{c}|^2$. These signs bring the present treatment very close to that in [3], although not identical.

2.2 The variations

At this point, one needs to take variations. Following [3] variations of the Lagrangian labels imply the following expressions for other variations

$$\delta \mathbf{X}_i = \partial_t \mathbf{Y}_i + [\mathbf{Y}_i, \mathbf{X}_i], \quad \delta \tilde{\mathbf{X}}_e = \partial_t \tilde{\mathbf{Y}}_e + [\tilde{\mathbf{Y}}_e, \tilde{\mathbf{X}}_e] + [\mathbf{X}_{\mathbf{V}_e}, \tilde{\mathbf{Y}}_e] - [\mathbf{X}_{\mathbf{W}}, \tilde{\mathbf{X}}_e] \quad (8)$$

$$\delta f_i = -\nabla \cdot (f_i \mathbf{Y}_i), \quad \delta \tilde{f}_e = -\nabla \cdot (\tilde{f}_e \tilde{\mathbf{Y}}_e) - \nabla \cdot (\tilde{f}_e \mathbf{X}_{\mathbf{W}}) \quad (9)$$

where $[\mathbf{P}, \mathbf{R}] = (\mathbf{P} \cdot \nabla) \mathbf{R} - (\mathbf{R} \cdot \nabla) \mathbf{P}$ is the vector field commutator, while $\mathbf{Y}_i(\mathbf{x}, \mathbf{v}, t)$, $\tilde{\mathbf{Y}}_e(\mathbf{x}, \mathbf{c}, t)$ and $\mathbf{W}(\mathbf{x}, t)$ are arbitrary vector fields vanishing at t_1 and t_2 . Here the vector field $\mathbf{X}_{\mathbf{V}_e}$ is constructed from \mathbf{V}_e as $\mathbf{X}_{\mathbf{V}_e} = (\mathbf{V}_e, \mathbf{c} \cdot \nabla_{\mathbf{x}} \mathbf{V}_e)$ and analogously for $\mathbf{X}_{\mathbf{W}}$. Notice that, upon introducing the absolute velocity $\mathbf{v} := \mathbf{c} + \mathbf{V}_e$, we may write by a slight abuse of notation

$$(\dot{\mathbf{x}}, \dot{\mathbf{v}})_e = (\tilde{\mathbf{u}}_e, \tilde{\mathbf{a}}_e) + (\mathbf{V}_e, \mathbf{c} \cdot \nabla \mathbf{V}_e) = \tilde{\mathbf{X}}_e + \mathbf{X}_{\mathbf{V}_e}$$

The physical interpretation for the emergence of these expressions goes back to the fact that $\tilde{\mathbf{X}}_e$ is a phase-space vector field that is expressed in the frame moving with \mathbf{V}_e ; see also [9], where the same method is applied to hybrid MHD models. Consequently, once variations (8) are taken in (6),

expanding the terms in $\delta \mathbf{a}_i$ and $\delta \tilde{\mathbf{a}}_e$ yields (7), while expanding the terms in $\delta \mathbf{u}_i$ and $\delta \tilde{\mathbf{u}}_e$ yields the expressions of the accelerations \mathbf{a}_i and $\tilde{\mathbf{a}}_e$. Eventually, one finds

$$\mathbf{X}_i = \left(\mathbf{v}, \frac{q_i}{m_i} \left[(\mathbf{v} - \mathbf{V}_e) \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \right) \quad (10)$$

$$\tilde{\mathbf{X}}_e = \left(\mathbf{c}, \mathbf{c} \times \left(\frac{q_e}{m_e} \mathbf{B} - \nabla \times \mathbf{V}_e \right) - 2\mathbf{c} \cdot \nabla \mathbf{V}_e + \frac{1}{m_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right) \quad (11)$$

In the next stage, following [10], arbitrary variations $(\delta\varphi, \delta\mathbf{A})$ yield the first two equations in (4), i.e. Ampere's law and charge neutrality

$$\nabla \times \mathbf{B} = n_e \mathbf{V}_e + \int \mathbf{v} f_i d^3\mathbf{v}, \quad \text{and} \quad n_e = - \int f_i d^3\mathbf{v} \quad (12)$$

On the other hand, variations of \mathbf{V}_e are given by

$$\delta \mathbf{V}_e = \partial_t \mathbf{W} + [\mathbf{W}, \mathbf{V}_e]$$

At this point, one makes the crucial *assumption of inertialess electrons* in the mean velocity equation: this step amounts exactly to letting

$$\epsilon \rightarrow 0,$$

in the action functional (6), so that the electron mean flow contributions to the kinetic energy is neglected, while retaining its current contribution in Ampere's law (12). The limit $\epsilon \rightarrow 0$ in the action (6) yields the following expression for the electric field \mathbf{E}

$$\mathbf{E} = - \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) = -\mathbf{V}_e \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e, \quad (13)$$

Ordinarily, a choice of gauge would be necessary to specify the dynamics of \mathbf{A} . For example, in [10] the hydrodynamic gauge $\varphi + \mathbf{V}_e \cdot \mathbf{A} = 0$ is chosen. However, taking the curl of the above equation yields (3). Notice that setting $\epsilon = 1$ in (6) amounts to including the electron inertial terms $\partial_t \mathbf{V}_e + (\mathbf{V}_e \cdot \nabla) \mathbf{V}_e$ in the right hand side of the second equality above.

Finally, following [7, 3, 10], one recalls that (6) can be expressed in terms of Lagrangian labels and derives the kinetic equations

$$\frac{\partial f_i}{\partial t} + \nabla \cdot (f_i \mathbf{X}_i) = 0, \quad \frac{\partial \tilde{f}_e}{\partial t} + \nabla \cdot (\tilde{f}_e \tilde{\mathbf{X}}_e) + \nabla \cdot (\tilde{f}_e \mathbf{X}_{\mathbf{V}_e}) = 0$$

by taking the time derivative of the Lagrange-to-Euler map for the two probability densities. Then, using the expressions (10)-(11), one obtains the kinetic equations (1) and (2).

3 Electron pressure dynamics

3.1 Hybrid model with frozen-in electron pressure

From the electron kinetics, one derives the evolution of the electron pressure tensor, which reads

$$\begin{aligned} \frac{\partial \tilde{\mathbb{P}}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) \tilde{\mathbb{P}}_e + (\nabla \cdot \mathbf{V}_e) \tilde{\mathbb{P}}_e + \tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e + (\tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e)^T \\ + \left(\frac{q_e}{m_e} \mathbf{B} - \boldsymbol{\omega}_e \right) \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \left(\frac{q_e}{m_e} \mathbf{B} - \boldsymbol{\omega}_e \right) = -\nabla \cdot \mathbf{Q}_e, \end{aligned}$$

where $\mathbf{Q}_e = \int \mathbf{c} \mathbf{c} \tilde{f}_e d^3\mathbf{c}$ is the heat flux tensor.

According to the above equation, the main problem concerning fully kinetic models for collisionless reconnection is that solving for two kinetic equations (ions and electrons) presents outstanding

computational difficulties [15] arising from the emergence of infinite moment hierarchies. This has led various authors to formulations of hybrid models, in which electron dynamics could be described by some kind of moment closure that retains the pressure tensor dynamics [5]. In particular, see also [19, 22, 13, 14, 21] and following papers by the same authors. In these works, the evolution of the electron pressure tensor dynamics is obtained by dropping the $\boldsymbol{\omega}_e$ - and \mathbf{Q}_e -terms and upon replacing the \mathbf{B} -terms as follows:

$$\frac{\partial \tilde{\mathbb{P}}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) \tilde{\mathbb{P}}_e + (\nabla \cdot \mathbf{V}_e) \tilde{\mathbb{P}}_e + \tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e + (\tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e)^T = -\frac{q_e |\mathbf{B}|}{m_e \tau} (\tilde{\mathbb{P}}_e - p_e \mathbb{I}),$$

where p_e is the scalar electron pressure and \mathbb{I} is the identity matrix, while τ is a phenomenological time-scale introduced *ad hoc* to match the available data. Increasing values of τ yield higher reconnected flux, although at some point singularities start to develop [22]. The above evolution equation yields reasonable results in many situations, although its underlying fundamental properties are not completely understood. Moreover, neglecting the heat flux tensor term may lead to physical inconsistencies [13], while a detailed study of its contribution is found in [14]. Still, most hybrid models for collisionless reconnection tend to disregard heat flux contributions.

If we accept that the heat flux can be neglected $\nabla \cdot \mathbf{Q}_e \simeq 0$, then, the model (1)-(3) (which retains the Coriolis force terms $\boldsymbol{\omega}_e \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \boldsymbol{\omega}_e$) yields the electron pressure dynamics in section 1.2. Moreover, in the strong electron magnetization limit $\mathbf{B} \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \mathbf{B} \simeq 0$, the electron pressure tensor becomes *frozen* into the electron mean velocity, so that the electron velocity correlations are constant along the electron mean flow. When this happens, one can simply replace the kinetic equation (1) by (5). This yields a hybrid model in which ion kinetics (2) is coupled to an electron velocity \mathbf{V}_e transporting its own pressure tensor $\tilde{\mathbb{P}}_e$, along the lines of [19, 22, 13, 14, 21]. More explicitly, this procedure yields

$$\frac{\partial \tilde{\mathbb{P}}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) \tilde{\mathbb{P}}_e + (\nabla \cdot \mathbf{V}_e) \tilde{\mathbb{P}}_e - (\boldsymbol{\omega}_e \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \boldsymbol{\omega}_e) + \tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e + (\tilde{\mathbb{P}}_e \cdot \nabla \mathbf{V}_e)^T = 0 \quad (14)$$

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left[(\mathbf{v} - \mathbf{V}_e) \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (15)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V}_e \times \mathbf{B} - \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right) \quad (16)$$

where

$$q_e n_e = -q_i \int f_i d^3 \mathbf{v}, \quad q_e n_e \mathbf{V}_e = \nabla \times \mathbf{B} - q_i \int \mathbf{v} f_i d^3 \mathbf{v}, \quad \tilde{\mathbb{P}}_e = m_e \int \mathbf{c} \mathbf{c} \tilde{f} d^3 \mathbf{c}. \quad (17)$$

Remark 1 (Strong electron magnetization limit) *The strong electron magnetization limit [21]*

$$\mathbf{B} \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \mathbf{B} \simeq 0$$

is recovered in the present approach by neglecting the minimal coupling term

$$q_e \int \mathbf{A} \cdot \tilde{\mathbf{u}}_e d^3 \mathbf{x} d^3 \mathbf{c}$$

in the action (6), which then assumes that fluctuations are not directly coupled to the magnetic field.

We point out that the Coriolis force terms $\boldsymbol{\omega}_e \times \tilde{\mathbb{P}}_e - \tilde{\mathbb{P}}_e \times \boldsymbol{\omega}_e$ are strictly necessary for a frozen-in electron pressure. Indeed, dropping these terms completely breaks this frozen-in condition. On the other hand, we should emphasize that this result is obtained by neglecting the heat flux contribution, which may not always be justifiable. The available data depend mainly on the particular situation that is being considered each time and no general statement is available in this regard.

3.2 Models with scalar electron pressure

It is well known that a purely scalar electron pressure is not sufficient to generate reconnection. On the other hand, a scalar electron pressure is still used in many situations involving the generation of energetic particles. Examples are given by Field Reversed Configuration devices and certain space plasmas [16, 19]. In these contexts, hybrid models can be formulated, as described in [19].

When the kinetic features of electron dynamics are neglected and one is interested only in the quantities \mathbf{V}_e and n_e , one only needs to replace the electron pressure tensor contribution $\nabla \cdot \tilde{\mathbb{P}}_e$ in (2)-(3) by the pressure gradient ∇p_e , as it is given in terms of the electron density n_e by an appropriate equation of state. The resulting model

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left[(\mathbf{v} - \mathbf{V}_e) \times \mathbf{B} + \frac{1}{q_e n_e} \nabla p_e \right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (18)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{V}_e) = 0 \quad (19)$$

was also analyzed in [18] from the Hamiltonian viewpoint. Here, the scalar pressure arises from an internal energy function $e(n_e)$ as $p_e := n_e^2 e'(n_e)$.

4 Stationary equations and constants of motion

In this section, we shall discuss a few properties of the stationary solutions of the model (1)-(3). While a nonlinear stability analysis requires knowledge of constants of motion that are not available a priori, some insight can be provided by the linearized equations. However, given the level of difficulty of the latter, this topic is left for further research in this direction. In this section, we shall make some comments on how the system (1)-(3) possesses Harris' current sheet solution and we shall show how the available constants of motion are insufficient for studying equilibria with non-zero current.

4.1 Stationary solutions and Harris' current sheets

Let us start our discussion by writing the stationary equations corresponding to (1)-(3). One has

$$(\mathbf{c} + \mathbf{V}_e) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{x}} + \left[\mathbf{c} \times \left(\frac{q_e}{m_e} \mathbf{B} - \nabla \times \mathbf{V}_e \right) - \mathbf{c} \cdot \nabla \mathbf{V}_e + \frac{1}{m_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{c}} = 0 \quad (20)$$

$$\mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left[(\mathbf{v} - \mathbf{V}_e) \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (21)$$

$$\nabla \times \left(\mathbf{V}_e \times \mathbf{B} - \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right) = 0 \quad (22)$$

with

$$q_e n_e = -q_i \int f_i d^3 \mathbf{v}, \quad q_e n_e \mathbf{V}_e = \nabla \times \mathbf{B} - q_i \int \mathbf{v} f_i d^3 \mathbf{v}. \quad (23)$$

Now, following Harris' work [6], we look for stationary states such that $\mathbf{E} = 0$. Then, upon recalling (13), we have

$$\mathbf{V}_e \times \mathbf{B} = \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e$$

and we observe that the electron solution has the general form

$$\tilde{f}_e = \tilde{f}_e \left(\frac{1}{2} |\mathbf{c} + \mathbf{V}_e|^2 - \frac{1}{2} |\mathbf{V}_e|^2 \right) = f_e \left(\frac{1}{2} |v|^2 - \frac{1}{2} |\mathbf{V}_e|^2 \right)$$

where we introduce $\mathbf{v} := \mathbf{c} + \mathbf{V}_e$. The above solution can be verified by replacing the above expression into equation (20), which may be written in terms of the canonical Poisson bracket $\{\cdot, \cdot\}$ as

$$\left\{ \tilde{f}_e, \frac{1}{2}|\mathbf{c}|^2 + \mathbf{c} \cdot \mathbf{V}_e \right\} + \frac{q_e}{m_e}(\mathbf{c} + \mathbf{V}_e) \times \mathbf{B} \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{c}} = 0.$$

Notice that $f_e(\mathbf{x}, \mathbf{v})$ and $f_i(\mathbf{x}, \mathbf{v})$ satisfy

$$\mathbf{v} \cdot \frac{\partial f_e}{\partial \mathbf{x}} + \left(\frac{q_e}{m_e} \mathbf{v} \times \mathbf{B} - \mathbf{v} \times \nabla \times \mathbf{V}_e + \nabla \mathbf{V}_e \cdot \mathbf{V}_e \right) \cdot \frac{\partial f_e}{\partial \mathbf{v}} = 0 \quad (24)$$

$$\mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0, \quad (25)$$

where the terms in \mathbf{V}_e appearing in the electron equation are produced by the assumption of negligible inertia of the mean flow.

Again, following [6], let us now restrict to the case when \mathbf{V}_e and \mathbf{V}_i are spatially constant and $\mathbf{V}_e = -\mathbf{V}_i$. Then, we obtain the usual Vlasov equations

$$\mathbf{v} \cdot \frac{\partial f_e}{\partial \mathbf{x}} + \frac{q_e}{m_e} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial f_e}{\partial \mathbf{v}} = 0 \quad (26)$$

$$\mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (27)$$

which, together with $q_e = e = -q_i$ and

$$n_e = \int f_i d^3 \mathbf{v}, \quad 2n_e \mathbf{V}_e = \nabla \times \mathbf{B},$$

lead to the Harris current sheet solution in [6].

Remark 2 (Static equilibria) Notice that for static equilibria in which $\mathbf{V}_e = \int f_i \mathbf{v} d^3 \mathbf{v} = 0$, one has

$$\mathbf{c} \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{x}} + \left[\frac{q_e}{m_e} \mathbf{c} \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{c}} = 0 \quad (28)$$

$$\mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left[\mathbf{v} \times \mathbf{B} + \frac{1}{q_e n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (29)$$

$$\nabla \times \left(\frac{1}{n_e} \nabla \cdot \tilde{\mathbb{P}}_e \right) = 0 \quad (30)$$

and if we assume $\mathbf{E} = (q_e n_e)^{-1} \nabla \cdot \tilde{\mathbb{P}}_e = 0$, we have

$$\mathbf{c} \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{x}} + \frac{q_e}{m_e} \mathbf{c} \times \mathbf{B} \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{c}} = 0 \quad (31)$$

$$\mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (32)$$

along with the zero-current relation $\nabla \times \mathbf{B} = 0$. Here the equation (31) is different from (26), since the first involves only the fluctuation velocity.

4.2 Total energy and constants of motion

In the search for new physical models, it is essential that the energy is an exact constant of motion. For example, many models that are derived by making assumptions directly in the equations of motion do not possess this vital property (see [18] for a similar discussion on hybrid MHD models). In the case of system (1)-(3), exact conservation of energy is guaranteed by the underlying variational principle

(6). When this happens, the system is Hamiltonian, which means it conserves energy and it possesses a Poisson bracket structure. The latter property can be verified explicitly also in the present context, although this requires cumbersome calculations, which in turn do not add anything to the physical content of equations (1)-(3). Therefore the explicit identification of the Poisson bracket structure underlying (1)-(3) is left for future developments. Here, we shall focus only on the constants of motion.

The following expression of the total energy remains constant under the dynamics (1)-(3):

$$\mathcal{E}(f_i, \tilde{f}_e, \mathbf{A}) = \frac{m_i}{2} \int f_i |\mathbf{v}|^2 d^3\mathbf{x} d^3\mathbf{v} + \frac{m_e}{2} \int \tilde{f}_e |\mathbf{c}|^2 d^3\mathbf{x} d^3\mathbf{c} + \frac{1}{2} \int |\nabla \times \mathbf{A}|^2 d^3\mathbf{x}, \quad (33)$$

as it can be easily checked by verifying that $\dot{\mathcal{E}} = 0$ explicitly. Notice that this is not the usual energy for the kinetic description of a hot plasma, mainly because only the fluctuation velocity $\mathbf{c} = \mathbf{v} - \mathbf{V}_e$ enters in the expression of \mathcal{E} .

Other constants of motion are present in the dynamics (1)-(3). Actually, these are two separate families of constants that are defined as follows:

$$C_i = \int \Phi_i(f_i) d^3\mathbf{x} d^3\mathbf{v}, \quad C_e = \int \Phi_e(\tilde{f}_e) d^3\mathbf{x} d^3\mathbf{v}$$

where Φ_i and Φ_e are arbitrary functions of their arguments.

Remark 3 (Energy-Casimir method for nonlinear stability) *Following [8], one can use the constants of motion to perform a nonlinear stability analysis for the system (1)-(3). In particular, this method identifies the equilibria by setting*

$$\delta(\mathcal{E} + C_i + C_e) = 0 \quad (34)$$

and gives Lyapunov stability whenever

$$\delta^2(\mathcal{E} + C_i + C_e) > 0 \quad (35)$$

However, the success of this method depends on the number of constants of motion that are available. It turns out that the three constants of motion found in this section restrict to consider only equilibria with zero current, thereby eliminating Harris' solution from the treatment. Indeed, (34) yields $\nabla \times \mathbf{B} = 0$ at the equilibrium. At this point, one can try to find new constants of motion to enrich the stability analysis. On the other hand, this task can be difficult to reach and it requires finding the explicit Poisson bracket for the system (1)-(3). This will be subject of future work.

5 Conclusions

This paper has presented the new fully kinetic model (1)-(3) for collisionless reconnection, whose main novelty is the introduction of a Coriolis force term

$$-\mathbf{c} \times \nabla \times \mathbf{V}_e \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{c}}$$

in the electron kinetic equation. Unlike previous models, this term was obtained by neglecting the inertia of the electron mean flow in the variational principle underlying the Maxwell-Vlasov system. Perhaps non surprisingly, the Coriolis force arises from a change of reference, similarly to the results in [17].

The principal consequence of introducing the Coriolis force is that neglecting the heat flux contribution in the electron pressure tensor dynamics yields a *frozen-in condition* for the electron pressure

tensor itself, in presence of a strong electron magnetization. In turn, this frozen-in condition can be used to formulate new hybrid models, along the lines of [13] and subsequent papers (analogue hybrid models were also presented in [5]). The frozen-in condition in resulting hybrid models would confer the electron pressure an intrinsic Lagrangian meaning, which would open various possibilities concerning simulation techniques.

The last part of the paper presented general considerations about the stationary equations for (1)-(3). Various specializations were considered and it was shown how the stationary equations comprise Harris' solution of the Maxwell-Vlasov system. Moreover, the explicit expression of the conserved total energy was provided, along with two more families of constants of motion. The question of how constants of motion can be used to perform a nonlinear stability analysis is left for future studies. This requires further constants that hopefully can be found from the explicit form of the Poisson bracket underlying (1)-(3). This Poisson bracket is the main object that underlies the Casimir method for Lyapunov stability and its identification can be realized by cumbersome computations in the Hamiltonian framework.

Acknowledgements

The author is indebted with Darryl Holm, Giovanni Lapenta, Philipp Morrison and Emanuele Tassi for many helpful discussions on these and related topics.

References

- [1] Arnold, V.I. *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*. Ann. Inst. Fourier (Grenoble) 16 (1966), 319–361.
- [2] Brizard, A.J. *New variational principle for the Vlasov-Maxwell equations* Phys. Rev. Lett. 84 (2000), no. 25, 5768–5771
- [3] Cendra, H.; Holm, D.D.; Hoyle, M.J.W.; Marsden, J.E. *The Maxwell-Vlasov equations in Euler-Poincaré form*. J. Math. Phys. 39 (1998), no. 6, 3138–3157
- [4] Dewar, R. *A Lagrangian theory for nonlinear wave packets in a collisionless plasma* J. Plasma Phys. 7 (1972), no. 2, 267–284
- [5] Divin, A.; Markidis, S.; Lapenta, G.; Semenov, V. S.; Erkaev, N.V.; Biernat, H.K. *Model of electron pressure anisotropy in the electron diffusion region of collisionless magnetic reconnection*. Phys. Plasmas 17 (2010), 122102
- [6] Harris, E.G. *On a Plasma sheath separating regions of oppositely directed magnetic field*. Nuovo Cimento 23 (1962), 115–121
- [7] Holm D. D., Marsden, J. E., and Ratiu, T. S. *The Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math. 137 (1998), 1–81.
- [8] Holm, D.D.; Marsden, J.E.; Ratiu, T.S.; Weinstein, A. *Nonlinear stability of fluid and plasma equilibria*. Phys. Rep. 123 (1985), no. 1-2, 1–116
- [9] Holm, D.D.; Tronci, C. *Euler-Poincaré formulation of hybrid plasma models*. Comm. Math. Sci. 10 (1), 191–222 (2012)
- [10] Ilgisonis, V.I.; Lakhin, V.P. *Lagrangian structure of hydrodynamic plasma models and conservation laws*. Plasma Phys. Rep. 25, no. 1, 1999, 58–69

- [11] Littlejohn, R.G. *Variational principles of guiding centre motion*. J. Plasma Phys. 29 (1983), no. 1, 111–125
- [12] Low, F.E. *A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas*. Proc. R. Soc. London, Ser. A 248 (1958), 282–287
- [13] Kuznetsova, M.M.; Hesse, M.; Winske, D. *Toward a transport model of collisionless magnetic reconnection*. J. Geophys. Res. 105 (2000), no. A4, p. 7601–7616
- [14] Kuznetsova, M.M.; Hesse, M.; Birn, J. *The role of electron heat flux in guide-field magnetic reconnection*. Phys. Plasmas 11 (2004), no. 12, 5387–5397
- [15] Markidis, S.; Lapenta, G.; Rizwan-uddin *Multi-scale simulations of plasma with iPIC3D*. Math. Comp. Sim. 80 (2010), 15091519
- [16] Park, W.; Belova, E.V.; Fu, G.Y.; Tang, X.Z.; Strauss, H.R.; Sugiyama, L.E. *Plasma simulation studies using multilevel physics models*. Phys. Plasmas 6 (1999), no. 6, 1796–1803.
- [17] Thyagaraja, A.; McClements, K. G. *Plasma physics in noninertial frames*, Phys. Plasmas 16 (2009), 092506
- [18] Tronci, C. *Hamiltonian approach to hybrid plasma models*, J. Phys. A: Math. Theor. 43 (2010), 375501
- [19] Winske, D.; Yin, L.; Omid, N.; Karimbadi, H.; Quest, K. *Hybrid simulation codes: past, present and future – a tutorial*. Lect. Notes Phys. 615 (2003), 136–165
- [20] Ye, H.; Morrison, P.J. *Action principles for the Vlasov equation*. Phys. Fluids B 4 (1992), no. 4, 771–777
- [21] Yin, L.; Winske, D. *Plasma pressure tensor effects on reconnection: hybrid and Hall-magnetohydrodynamics simulations* Phys. Plasmas 10 (2003), no. 5, 1595–1604
- [22] Yin, L.; Winske, D.; Gary, S.P.; Birn, J. *Hybrid and Hall-MHD simulations of collisionless reconnection: dynamics of the electron pressure tensor*. J. Geophys. Res. 106 (2006), no. A6, p. 10761–10776